JOURNAL OF APPROXIMATION THEORY **17**, 150–165 (1976)

The Dominated Integral

CHARLES F. OSGOOD

Naval Research Laboratory, Washington, D. C. 20375

AND

OVED SHISHA

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881

Received April 22, 1974

I. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we introduce a new concept of integral on $(0, 1]$ (the "dominated integral") intimately connected with the problem of numerical integration of unbounded functions. The existence of the dominated integral I of a function f implies the convergence of the improper Riemann integral $\int_{0+}^{1} f(x)\,dx$, and its equality to I, but $\int_{0+}^{1} f(x)\,dx$ may converge without existence of I. An important difference between the two concepts is that while $\int_{0+}^{1} f(x)\,dx$ is defined as an iterated limit (i.e., the limit of a proper Riemann integral, itself a limit), the dominated integral is defined as a single limit.

Recently, a concept similar to that of the dominated integral was introduced, concerning integration over $[0, \infty)$ (the "simple integral," see [1, 2]). However, the dominated integral and the simple integral appear to be of somewhat different nature: The first is an absolute integral, the second is not. In fact, one can readily see that if one tries to imitate the definition of simple integrability by replacing $[0, \infty)$ with $(0, 1]$, and ∞ by 0, one obtains essentially the concept of (proper) Riemann integrability on $[0, 1]$.

The theory of the dominated integral is strongly related to the problem: Under what conditions can quadrature formulas effective for Riemann integrable functions on $[0, 1]$ be used for the numerical evaluation of improper Riemann integrals $\int_{0+}^{1} f(x)\,dx$? The theoretical study of this type of question was initiated by Davis and Rabinowitz [3], and was followed by further work [4–7]. As the practical use of quadrature formulas to compute improper Riemann integrals without a theoretical justification has become quite common, the need for such a theoretical study is unquestionable.

It turns out that, for a function f on $(0, 1]$, the existence of its dominated integral is a necessary and sufficient condition for f to be improperly Riemann integrable there, and to satisfy $\lim_{n\to\infty} \Phi_n^*(f) = \int_{0+}^1 f(x)\,dx$ for every sequence $(\Phi_n^*)_{n=1}^\infty$ of quadrature formulas of a very general type. This is shown in a subsequent article [8]. Here we only mention two applications from [8]:

I. Suppose $(R_n(f))_{n=1}^\infty$ is a sequence of compound rules on $[0, 1]$ not involving $f(0)$, and integrating 1 exactly, namely,

$$R_n(f) \equiv \sum_{k=1}^{n} \sum_{r=1}^{m} w_r n^{-1} f((k-1+x_r)\,n^{-1}), \qquad n = 1, 2, \ldots,$$

where $w_1, \ldots, w_m$ are given complex constants with $\sum_{j=1}^m w_j = 1$, and $0 < x_1 < \cdots < x_m \leqslant 1$. Then $\lim_{n\to\infty} R_n(f) = \int_{0+}^1 f(x)\,dx$ for every f whose dominated integral exists.

II. One can define the dominated integral on any interval $(a, b]$, $-\infty < a < b < \infty$. Let $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$, $-\frac{1}{2} \leqslant \beta \leqslant \frac{1}{2}$, and, for $n = 1, 2, \ldots$, let

$$Q_n(F) \equiv \sum_{k=1}^{n} w_{n,k} F(x_{n,k})$$

be the n-point Gauss–Jacobi quadrature formula corresponding to the weight function $w(x) \equiv (1-x)^\alpha(1+x)^\beta$. If the dominated integral of a function f on $(-1, 1]$ exists, then $\lim_{n\to\infty} Q_n(f/w) = \int_{-1+}^1 f(x)\,dx$.

We now define the dominated integral, state its fundamental properties and relate it to the "simple integral" of [1, 2].

DEFINITION 1. Let f be a complex function on $(0, 1]$. A dominated integral of f is a number $I(f)$ having the property: For each $\epsilon > 0$ there exist δ and χ, $0 < \delta < 1$, $0 < \chi < 1$, such that

$$\left| I(f) - \sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}) \right| < \epsilon \tag{1}$$

whenever $0 < t_0 < t_1 < \cdots < t_n = 1$, $t_0 < \chi$, $t_{j-1} \leqslant \tau_j \leqslant t_j$, and $t_{j-1}t_j^{-1} > 1 - \delta$, $j = 1, 2, \ldots, n$. (The justification for the adjective "dominated" will be clear from Theorem 3.) Dominant integrability of f means existence of such an $I(f)$. It is not difficult to see that if an $I(f)$ exists, it is unique.

THEOREM 1. *If a dominated integral of f exists, then f is improperly Riemann integrable on* $(0, 1]$, *and* $I(f) = \int_{0+}^1 f(t)\,dt$, *the improper Riemann integral of f on* $(0, 1]$.

(By the improper Riemann integral of $f(t)$ on $(0, 1]$[$g(x)$ on $[0, \infty)$] we mean $\lim_{\epsilon\to 0+} \int_\epsilon^1 f(t)\,dt[\lim_{R\to\infty} \int_0^R g(x)\,dx]$, assumed to be finite, where for each ϵ, $0 < \epsilon < 1$, [R, $R > 0$] $\int_\epsilon^1 f(t)\,dt[\int_0^R g(x)\,dx]$ is assumed to exist as a proper Riemann integral.)

If f is a bounded complex function on $[a, b]$ we shall denote by $w(f, a, b)$ the oscillation of f on $[a, b]$, i.e., the supremum of the set of all $|f(t_1) - f(t_2)|$ with $a \leqslant t_1 \leqslant t_2 \leqslant b$.

Given a complex function $f(t)$ defined and bounded on each closed subinterval of $(0, 1]$, and any sequence $0 < t_0 < t_1 < \cdots < t_n = 1$, let $\mathrm{OS}(f; t_0, \ldots, t_n)$ denote the oscillation sum

$$\sum_{j=1}^{n} w(f, t_{j-1}, t_j)(t_j - t_{j-1}).$$

DEFINITION 2. A complex function f satisfies the Riemann condition for the dominated integral (RCDI) if and only if the following two conditions hold:

(i) f is defined on $(0, 1]$, and bounded on each of its closed subintervals; and

(ii) for each $\epsilon > 0$ there exists δ, $0 < \delta < 1$, such that whenever $0 < t_0 < t_1 < \cdots < t_n = 1$, and $t_{j-1}t_j^{-1} > 1 - \delta$, $j = 1, \ldots, n$, we have $\mathrm{OS}(f; t_0, \ldots, t_n) < \epsilon$.

THEOREM 2. *Let f be a complex function on* $(0, 1]$. *The dominated integral of f exists if and only if f satisfies* RCDI.

COROLLARY 1. *If the dominated integral of f exists, then so does the dominated integral of $|f|$.*

Proof of Corollary 1. Since always $||f(t_1)| - |f(t_2)|| \leqslant |f(t_1) - f(t_2)|$, we see that if f satisfies RCDI, so does $|f|$.

COROLLARY 2. *If the dominated integral of f exists, then there is a real monotone decreasing (by which we always mean "nonincreasing") function $\hat{f}$ on* $(0, 1]$ *such that* (i) $\hat{f} \geqslant |f|$ *throughout* $(0, 1]$, *and* (ii) $\hat{f}$ *is improperly Riemann integrable on* $(0, 1]$.

Proof of Corollary 2. Set $\hat{f}(t) = \sup_{t \leqslant x \leqslant 1} |f(x)|$ $(0 < t \leqslant 1)$. (See Theorem 2 and Definition 2, (i)). Then throughout $(0, 1]$, $\hat{f} \geqslant |f|$, and $\hat{f}$ is monotone decreasing. Clearly $\hat{f}(t)$ is bounded on each closed subinterval of $(0, 1]$. We shall show that if $0 < a < b \leqslant 1$, then $w(\hat{f}, a, b) \leqslant w(|f|, a, b)$. It would then follow that $\hat{f}$ satisfies RCDI, and we would be through by

Theorem 2. We need *only consider the case* $\hat{f}(a) - \hat{f}(b) > 0$. In this case $\hat{f}(a) = \sup_{a \leqslant x \leqslant b} |f(x)|$. If $a \leqslant x_1 \leqslant x_2 \leqslant b$, then

$$\begin{aligned} |\hat{f}(x_1) - \hat{f}(x_2)| &= \hat{f}(x_1) - \hat{f}(x_2) \\ &\leqslant \hat{f}(a) - \hat{f}(b) \leqslant \hat{f}(a) - |f(b)| = \sup_{a \leqslant x \leqslant b} (|f(x)| - |f(b)|) \\ &\leqslant \sup_{a \leqslant x \leqslant x' \leqslant b} \big|\, |f(x)| - |f(x')|\, \big| = w(|f|, a, b). \end{aligned}$$

DEFINITION 3. A complex function f on $(0, 1]([0, \infty))$ is called absolutely dominantly integrable (absolutely simply integrable) if and only if f is Riemann integrable on each closed bounded subinterval of $(0, 1]([0, \infty))$, and $|f|$ is dominantly integrable (simply integrable).

Observe that simple integrability implies improper Riemann integrability on $[0, \infty)$ [2, p. 931].

DEFINITION 4. A complex function f is said to have property D (for "dominated") on $(0, 1]$, or on $[0, \infty)$, if and only if f is Riemann integrable on each closed bounded subinterval of $(0, 1]$, or of $[0, \infty)$, and if there exists a monotone decreasing improperly Riemann integrable function g on $(0, 1]$, or on $[0, \infty)$, such that at each point of the interval, $g(t) \geqslant |f(t)|$.

THEOREM 3. *The following are equivalent*: (i) *dominant integrability*; (ii) *absolute dominant integrability*; (iii) *property D on* $(0, 1]$; *and* (iv) *Riemann integrability on each closed subinterval of* $(0, 1]$ *along with domination of absolute value on* $(0, 1]$ *by some dominantly integrable function.*

(That (i) implies (ii) follows from Theorem 1 and Corollary 1. That (ii) implies (iii) follows from Corollary 2. By Corollary 2 applied to the dominating function, we see that (iv) implies (iii). That (i) implies (iv) is seen by Theorem 1 and Corollary 1, letting the absolute value of the function dominate itself. Thus it merely remains to prove that (iii) implies (i).)

THEOREM 4. *Absolute simple integrability implies but is not implied by simple integrability and is implied by but does not imply property D on* $[0, \infty)$.

THEOREM 5. *Dominant integrability of a function f is equivalent to absolute simple integrability of* $f(e^{-x})\, e^{-x}$, *but does not imply property D of* $f(e^{-x})\, e^{-x}$ *on* $[0, \infty)$.

Let G denote the set of all real functions g with domain $[0, \infty)$, with g' continuous and negative on $[0, \infty)$($g'(0)$ being a right-hand derivative), $g(0) = 1$, and $\lim_{x\to\infty} g(x) = 0$. For each g in G, let S_g be the set of all complex functions f with domain $(0, 1]$ for which $f(g)\, g'$ is simply integrable.

THEOREM 6. (i) *The class S of dominantly integrable functions with domain* (0, 1] *does not equal any* S_g. (ii) *Given* $g \in G$, *there is an improperly Riemann integrable function on* (0, 1] *which is not in* S_g. (iii) *Given any f with domain* (0, 1] *which is* > 0, *continuous and improperly Riemann integrable on* (0, 1], $f \in S_g$ *for some g in G, and* $f \notin S_g$ *for some other g in G.*

The next theorem shows that if we drop the condition $g'(x) < 0$ from the definition of G, then part (i) of Theorem 6 becomes false.

THEOREM 7. *There exists a real function* $g(x)$ *with* g' *continuous in* $[0, \infty)$, $g(0) = 1$, $\lim_{x\to\infty} g(x) = 0$, *and* $0 < g(x) \leqslant 1$ *throughout* $[0, \infty)$ *such that, given any complex function f on* (0, 1], $f(g)\, g'$ *is simply integrable if and only if f is dominantly integrable.*

(Not too surprisingly, in light of Theorem 5, the $g(x)$ to be constructed in the proof of Theorem 7 is related to the function e^{-x}, without being monotone.)

II. PROOF OF THEOREMS 1–3

LEMMA 1. *Let f be dominantly integrable, and let* $0 < a < b \leqslant 1$. *Then f is Riemann integrable on* $[a, b]$.

Proof of Lemma 1. We shall prove that f is Riemann integrable on $[a, 1]$. As we shall see, it suffices to prove the following statement: For each $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $a = x_0 < x_1 < \cdots < x_n = 1$, $x_{j-1}x_j^{-1} > 1 - \delta_1$, $x_{j-1} \leqslant \xi_j \leqslant x_j$, and $x_{j-1} \leqslant \xi_j' \leqslant x_j$, $j = 1, 2,..., n$, then

$$\left| \sum_{j=1}^{n} [f(\xi_j) - f(\xi_j')](x_j - x_{j-1}) \right| < \epsilon. \tag{2}$$

Indeed, assume its truth. Let $\delta_2 = a\delta_1$. If $a = x_0 < x_1 < \cdots < x_n = 1$, and $x_j - x_{j-1} < \delta_2$, $j = 1, 2,..., n$, then $1 - x_{j-1}x_j^{-1} < a^{-1}\delta_2$, and $x_{j-1}x_j^{-1} > 1 - \delta_1$, so we have inequality (2) which is easily seen to imply that $\mathrm{OS}(\mathrm{Re}(f); x_0, x_1,..., x_n) \leqslant \epsilon$, and $\mathrm{OS}(\mathrm{Im}(f); x_0, x_1,..., x_n) \leqslant \epsilon$; hence, $f(t)$ is Riemann integrable on $[a, 1]$. What remains is to prove the above statement.

Choose δ and χ, both in (0, 1), so that any two sums of the type appearing in (1), with $t_0 < \chi$, and every $t_{j-1}t_j^{-1} > 1 - \delta$, differ in absolute value by less than ϵ. Set $\delta_1 = \delta$, and choose a positive integer N such that $(1 - \frac{1}{2}\delta)^N a < \chi$. Set $t_0 = \tau_0 = \tau_0' = (1 - \frac{1}{2}\delta)^N a,..., t_{N-1} = \tau_{N-1} = \tau_{N-1}' = (1 - \frac{1}{2}\delta)a$, and, for $0 \leqslant j \leqslant n$, set $t_{N+j} = x_j$, $\tau_{N+j} = \xi_j$, and $\tau_{N+j}' = \xi_j'$.

Then we have

$$\left| \sum_{j=0}^{n} [f(\xi_j) - f(\xi_j')](x_j - x_{j-1}) \right| = \left| \sum_{j=0}^{N+n} [f(\tau_j) - f(\tau_j')](t_j - t_{j-1}) \right| < \epsilon.$$

This completes the proof of Lemma 1.

LEMMA 2. *Let f be dominantly integrable. Then it satisfies RCDI.*

Proof of Lemma 2. Since f is Riemann integrable on each closed subinterval of $(0, 1]$, it is bounded on each such subinterval. Given $\epsilon > 0$, choose δ, χ in $(0, 1)$ such that (1) holds under the conditions following it. Let $0 < t_0 < t_1 < \cdots < t_n = 1$, $t_{j-1}t_j^{-1} > 1 - \delta$, $j = 1, 2,..., n$. Let $g = \text{Re}(f)$, $h = \text{Im}(f)$. Now $w(f, t_{j-1}, t_j) \leqslant w(g, t_{j-1}, t_j) + w(h, t_{j-1}, t_j)$; hence, if we show that $\text{OS}(g; t_0, ..., t_n)$ and $\text{OS}(h; t_0, ..., t_n)$ are each less than 2ϵ, then it would follow that $\text{OS}(f; t_0, ..., t_n) < 4\epsilon$. Consider, e.g., $\text{OS}(g; t_0, ..., t_n)$. Suppose, first, $t_0 < \chi$. If τ_j and τ_j' lie in $[t_{j-1}, t_j]$, $j = 1, ..., n$, then, using (1),

$$\left| \sum_{j=1}^{n} (g(\tau_j) - g(\tau_j'))(t_j - t_{j-1}) \right| \leqslant \left| \sum_{j=1}^{n} (f(\tau_j) - f(\tau_j'))(t_j - t_{j-1}) \right| < 2\epsilon.$$

Thus $\text{OS}(g; t_0, ..., t_n) = \sum_{j=1}^{n} w(g, t_{j-1}, t_j)(t_j - t_{j-1}) \leqslant 2\epsilon$. If $t_0 \geqslant \chi$, choose a positive integer N such that $(1 - \frac{1}{2}\delta)^N t_0 < \chi$. Then $\text{OS}(f; t_0, ..., t_n) \leqslant \text{OS}(f; (1 - \frac{1}{2}\delta)^N t_0, ..., (1 - \frac{1}{2}\delta) t_0, t_0, t_1, ..., t_n) < 4\epsilon$. This proves Lemma 2.

Recall the definition of $\hat{f}$ from the proof of Corollary 2.

LEMMA 3. *Suppose that f satisfies RCDI; then $\hat{f}$ is improperly Riemann integrable on* $(0, 1]$.

Proof of Lemma 3. For a proper δ, $0 < \delta < 1$, and for every positive integer N,

$$\sum_{j=0}^{N} [\hat{f}((1 - \tfrac{1}{2}\delta)^{j+1}) - \hat{f}((1 - \tfrac{1}{2}\delta)^j)](1 - \tfrac{1}{2}\delta)^j$$

$$\leqslant \sum_{j=0}^{N} w(f, (1 - \tfrac{1}{2}\delta)^{j+1}, (1 - \tfrac{1}{2}\delta)^j)(1 - \tfrac{1}{2}\delta)^j$$

$$= 2\delta^{-1} \sum_{j=0}^{N} w(f, (1 - \tfrac{1}{2}\delta)^{j+1}, (1 - \tfrac{1}{2}\delta)^j)((1 - \tfrac{1}{2}\delta)^j - (1 - \tfrac{1}{2}\delta)^{j+1}) < \delta^{-1},$$

by RCDI. Since $\hat{f}$ is monotone, it is Riemann integrable on each $[a, 1]$, $0 < a < 1$. Since $\hat{f} \geqslant 0$, it is improperly Riemann integrable on $(0, 1]$ if

$\int_\epsilon^1 f(t)\,dt$ is bounded for $0 < \epsilon < 1$. But an upper bound for this integral is

$$\left(\sum_{j=0}^{\infty} [f((1-\tfrac{1}{2}\delta)^{j+1}) - f((1-\tfrac{1}{2}\delta)^j)](1-\tfrac{1}{2}\delta)^j\right) + f(1) \leqslant \delta^{-1} + f(1).$$

LEMMA 4. *Let f satisfy RCDI, and let $0 < a < b \leqslant 1$. Then f is Riemann integrable on $[a, b]$.*

Proof of Lemma 4. For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\sum_{j=1}^{n} (f(\tau_j) - f(\tau_j'))(t_j - t_{j-1})\right| < \epsilon$$

if $a = t_0 < t_1 < \cdots < t_n = 1$, and if, for $j = 1, 2,\ldots, n$, both τ_j, τ_j' belong to $[t_{j-1}, t_j]$, and $t_{j-1}t_j^{-1} > 1 - \delta$. This, however, implies the Riemann integrability of f on $[a, 1]$; see the proof of Lemma 1.

LEMMA 5. *Suppose f is a complex function, Riemann integrable on each $[a, b] \subset (0, 1]$. For each ϵ, ϵ_1 ($\epsilon > 0$, $0 < \epsilon_1 < 1$) there exists δ_1 in $(0, \frac{1}{2})$ such that if $0 < t_0 < \cdots < t_n = 1$; $t_{j-1} \leqslant \tau_j \leqslant t_j$, $t_{j-1}t_j^{-1} > 1 - \delta_1$ ($j = 1, 2,\ldots, n$), and $t_{n_1-1} \leqslant \epsilon_1 < t_{n_1}$ for some n_1, $1 \leqslant n_1 \leqslant n$, then*

$$\left|\sum_{j=n_1}^{n} f(\tau_j)(t_j - t_{j-1}) - \int_{t_{n_1-1}}^{1} f(t)\,dt\right| < \epsilon.$$

Proof of Lemma 5. We have

$$\begin{aligned}&\left|\sum_{j=n_1}^{n} f(\tau_j)(t_j - t_{j-1}) - \int_{t_{n_1-1}}^{1} f(t)\,dt\right| \\ &\quad\leqslant \left|\sum_{j=n_1+1}^{n} f(\tau_j)(t_j - t_{j-1}) + f(\epsilon_1)(t_{n_1} - \epsilon_1) - \int_{\epsilon_1}^{1} f(t)\,dt\right| \\ &\quad\quad + 3M(t_{n_1} - t_{n_1-1}),\end{aligned}$$

where $M = f((1-\delta_1)\,\epsilon_1)$, and $\sum_{j=n_1+1}^{n} = 0$ if $n_1 = n$. Using the Riemann integrability of f on $[\epsilon_1, 1]$, Lemma 5 follows.

LEMMA 6. *Let a complex function f be Riemann integrable on each $[a, b] \subset (0, 1]$, and let f be improperly Riemann integrable on $(0, 1]$; then f is dominantly integrable and improperly Riemann integrable on $(0, 1]$, and $I(f)$ (see Definition 1) is the improper Riemann integral of f on $(0, 1]$.*

In light of Lemmas 2, 3, and 4, Lemma 6 implies Theorems 1 and 2.

Proof of Lemma 6. Since $\hat{f}$ is improperly Riemann integrable on $(0, 1]$, and f is Riemann integrable on each closed subinterval of $(0, 1]$, it follows that $\int_{0+}^{1} |f(t)|\, dt$ and $\int_{0+}^{1} f(t)\, dt$ exist. Given $\epsilon > 0$, choose ϵ_1, $0 < \epsilon_1 < 1$, such that $\int_{0+}^{\epsilon_1} \hat{f}(t)\, dt < \epsilon$. Using Lemma 5, we see that there exists δ, $0 < \delta < \frac{1}{2}$, such that if $0 < t_0 < \cdots < t_n = 1$; $t_{j-1} \leqslant \tau_j \leqslant t_j$, $t_{j-1}t_j^{-1} > 1 - \delta$ $(j = 1,\ldots, n)$, and $t_{n_1-1} \leqslant \epsilon_1 < t_{n_1}$ for some n_1, $1 \leqslant n_1 \leqslant n$, then

$$\left| \sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}) - \int_{0+}^{1} f(t)\, dt \right| < \left| \sum_{j=1}^{n_1-1} f(\tau_j)(t_j - t_{j-1}) \right| + \int_{0+}^{\epsilon_1} |f(t)|\, dt + \epsilon.$$

Now, under these conditions,

$$\begin{aligned} \left| \sum_{j=1}^{n_1-1} f(\tau_j)(t_j - t_{j-1}) \right| &\leqslant \sum_{j=1}^{n_1-1} \hat{f}(t_{j-1})(t_j - t_{j-1}) \\ &\leqslant \sum_{j=1}^{n_1-1} \hat{f}((1-\delta)\, t_j)(t_j - t_{j-1}) \\ &\leqslant \int_{0+}^{\epsilon_1} \hat{f}((1-\delta)\, t)\, dt \\ &\leqslant (1-\delta)^{-1} \int_{0+}^{\epsilon_1} \hat{f}(t)\, dt \\ &\leqslant 2 \int_{0+}^{\epsilon_1} \hat{f}(t)\, dt. \end{aligned}$$

Hence

$$\left| \sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}) - \int_{0+}^{1} f(t)\, dt \right| < 3 \int_{0+}^{\epsilon_1} \hat{f}(t)\, dt + \epsilon < 4\epsilon.$$

LEMMA 7. *Property D on* $(0, 1]$ *implies dominant integrability.*

The proof of Lemma 7 will complete that of Theorem 3.

Proof of Lemma 7. If there exists a monotone decreasing function $g \geqslant |f|$ which is improperly Riemann integrable on $(0, 1]$, then $\hat{f} \leqslant g$ is also improperly Riemann integrable on $(0, 1]$. Thus Lemma 7 follows from Lemma 6.

III. Proof of Theorems 4 and 5

Proof of Theorem 4. The first nonimplication was shown in [1, p. 7]. The second nonimplication is exemplified by the function f which is identically zero on $[0, \infty)$ except that $f(n^2) = n^{-2}$, $n = 1, 2, \ldots$. The ϵ-variation of this function is $\leqslant \frac{1}{6}\pi^2$ for each $\epsilon > 0$ (see [1, p. 9]), and the function is improperly Riemann integrable on $[0, \infty)$; hence, by [1, Theorem 3] it is simply (thus absolutely simply) integrable. On the other hand $\hat{f}(x) \equiv \sup_{t \geqslant x} f(t)$ is not improperly Riemann integrable on $[0, \infty)$, so $f(x)$ does not have property D on $[0, \infty)$.

For a proof of the second implication of the Theorem see [1, Theorem 3 and Example b on p. 9]. The first implication can be proved as follows: Let f be an absolutely simply integrable function. By the first part of Theorem 5, $f(x) \equiv g(e^{-x})\, e^{-x}$, where g is dominantly integrable. So (with $\hat{g}(t) \equiv \sup_{t \leqslant x \leqslant 1} |g(x)|$), $|f(x)| \leqslant \hat{g}(e^{-x})\, e^{-x}$ on $[0, \infty)$. Hence if $0 \leqslant x_0 < x_1 < \cdots < x_N < \infty$, $x_j - x_{j-1} \geqslant \epsilon > 0$, $j = 1, \ldots, N$, then (with $x_{N+1} = x_N + \epsilon$),

$$\begin{aligned}
\sum_{j=1}^{N} |f(x_j) - f(x_{j-1})| &\leqslant 2 \sum_{j=0}^{N} |f(x_j)| \\
&\leqslant 2 \sum_{j=0}^{N} \hat{g}(e^{-x_j})\, e^{-x_j} \\
&\leqslant 2(1 - e^{-\epsilon})^{-1} \sum_{j=0}^{N} \hat{g}(e^{-x_j})(e^{-x_j} - e^{-x_{j+1}}) \\
&\leqslant 2(1 - e^{-\epsilon})^{-1} \int_{0+}^{1} \hat{g}(t)\, dt < \infty.
\end{aligned}$$

Thus f is of BCV; hence, it is simply integrable [1, Theorem 3].

Proof of Theorem 5. Assuming the equivalence in Theorem 5, the non-implication there follows by the last part of Theorem 4. Dominant integrability of f implies, by Corollary 1, Theorem 1 and a change of variable, that $|f(e^{-x})|\, e^{-x}$ is improperly Riemann integrable on $[0, \infty)$. Also if $0 \leqslant x_0 < x_1 < \cdots < x_N < \infty$, and $x_j - x_{j-1} \geqslant \epsilon > 0$, $j = 1, 2, \ldots, N$, then

$$\begin{aligned}
\sum_{j=0}^{N} \hat{f}(e^{-x_j})\, e^{-x_j} &\leqslant (1 - e^{-\epsilon})^{-1} \Big[\sum_{j=1}^{N} \hat{f}(e^{-x_{j-1}})(e^{-x_{j-1}} - e^{-x_j}) \\
&\quad + \hat{f}(e^{-x_N})(e^{-x_N} - e^{-x_N - \epsilon}) \Big] \leqslant (1 - e^{-\epsilon})^{-1} \int_{0+}^{1} \hat{f}(t)\, dt < \infty.
\end{aligned}$$

We therefore have

$$\sum_{j=1}^{N} \big|\, |f(e^{-x_j})|\, e^{-x_j} - |f(e^{-x_{j-1}})|\, e^{-x_{j-1}} \big|$$

$$\leqslant \sum_{j=1}^{N} \big|\, |f(e^{-x_j})| - |f(e^{-x_{j-1}})| \,\big|\, e^{-x_j} + |f(e^{-x_{j-1}})|\, (e^{-x_{j-1}} - e^{-x_j})$$

$$\leqslant \sum_{j=1}^{N} 2\hat{f}(e^{-x_j})\, e^{-x_j} + \sum_{j=1}^{N} \hat{f}(e^{-x_{j-1}})\, e^{-x_{j-1}}$$

$$\leqslant 3 \sum_{j=0}^{N} \hat{f}(e^{-x_j})\, e^{-x_j}$$

$$\leqslant 3(1 - e^{-\epsilon})^{-1} \int_{0+}^{1} \hat{f}(t)\, dt.$$

Thus, using [3, Theorem 3], dominant integrability of f implies absolute simple integrability of $f(e^{-x})\, e^{-x}$.

Assuming absolute simple integrability of $f(e^{-x})\, e^{-x}$ we have (by a change of variable): (i) $f(x)$ is Riemann integrable on each $[a, b] \subset (0, 1]$; (ii) the improper Riemann integral $\int_{0+}^{1} |f(t)|\, dt$ exists; and (iii) for each δ, $0 < \delta < 1$, and every infinite sequence of positive numbers $1 \geqslant \xi_0 > \xi_1 > \xi_2 > \cdots$ with each $\xi_j \xi_{j-1}^{-1} \leqslant 1 - \delta$, $\sum_{j=1}^{\infty} \big|\, |f(\xi_j)|\, \xi_j - |f(\xi_{j-1})|\, \xi_{j-1} \big| < \infty$. If $\hat{f}$ is improperly Riemann integrable on (0, 1], then f has property D there; hence, f is dominantly integrable by Theorem 3. Therefore, we shall assume that $\hat{f}$ is not improperly Riemann integrable on (0, 1]. Choose δ in (0, 1). Then

$$\sum_{j=0}^{N} [\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^{j})](1 - \delta)^{j}$$

$$\geqslant -\hat{f}(1) + \sum_{j=0}^{N-1} \hat{f}((1 - \delta)^{j+1})[(1 - \delta)^{j} - (1 - \delta)^{j+1}]$$

$(n = 1, 2, \ldots)$, so that

$$\sum_{j=0}^{\infty} [\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^{j})](1 - \delta)^{j} = \infty.$$

For $j = 0, 1, 2, \ldots$, pick a τ_j, $(1 - \delta)^{j+1} \leqslant \tau_j \leqslant (1 - \delta)^{j}$, such that $|f(\tau_j)| \geqslant \frac{1}{2}[\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^{j})]$ (existence of such τ_j becomes evident on considering each of the cases: $\hat{f}((1 - \delta)^{j+1}) = \hat{f}((1 - \delta)^{j})$ and $\hat{f}((1 - \delta)^{j+1}) > \hat{f}((1 - \delta)^{j})$). Then, for $N = 1, 2, \ldots$,

$$\sum_{j=0}^{N} |f(\tau_j)|\, \tau_j \geqslant (1 - \delta)\, \tfrac{1}{2} \sum_{j=0}^{N} [\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^{j})](1 - \delta)^{j},$$

so that $\sum_{j=0}^{\infty} |f(\tau_j)| \tau_j = \infty$. Choose k, $0 \leqslant k \leqslant 3$, such that $\sum_{j=0}^{\infty} |f(\tau_{4j+k})| \times \tau_{4j+k} = \infty$, and set $\xi_{2j} = \tau_{4j+k}$, $j = 0, 1, 2,\ldots$. Next, for $j = 0, 1, 2,\ldots$, choose ξ_{2j+1}, $(1-\delta)^{4j+3+k} \leqslant \xi_{2j+1} \leqslant (1-\delta)^{4j+2+k}$, such that

$$|f(\xi_{2j+1})| \leqslant 1 + \inf_{(1-\delta)^{4j+3+k} \leqslant x \leqslant (1-\delta)^{4j+2+k}} |f(x)|.$$

Let N be an integer $\geqslant 0$. Then

$$\begin{aligned}
&\delta \sum_{j=0}^{N} |f(\xi_{2j+1})| (1-\delta)^{4j+2+k} \\
&\quad = \sum_{j=0}^{N} |f(\xi_{2j+1})| [(1-\delta)^{4j+2+k} - (1-\delta)^{4j+3+k}] \\
&\quad \leqslant \sum_{r=0}^{4N+2+k} [\inf_{(1-\delta)^{r+1} \leqslant x \leqslant (1-\delta)^r} \{|f(x)| + 1\}][(1-\delta)^r - (1-\delta)^{r+1}] \\
&\quad \leqslant \int_{0+}^{1} (|f(t)| + 1)\, dt;
\end{aligned}$$

hence, $\sum_{j=0}^{N} |f(\xi_{2j+1})| \xi_{2j+1} \leqslant \delta^{-1} \int_{0+}^{1} (|f(t)| + 1)\, dt$.

Since $\xi_j \xi_{j-1}^{-1} \leqslant 1 - \delta$, $j = 1, 2,\ldots$, we have

$$\begin{aligned}
\sum_{j=1}^{2N+2} \big|\, |f(\xi_j)| \xi_j - |f(\xi_{j-1})| \xi_{j-1} \big| &\geqslant \sum_{j=1}^{N+1} \big|\, |f(\xi_{2j})| \xi_{2j} - |f(\xi_{2j-1})| \xi_{2j-1} \big| \\
&\geqslant \sum_{j=1}^{N+1} |f(\xi_{2j})| \xi_{2j} - \sum_{j=1}^{N+1} |f(\xi_{2j-1})| \xi_{2j-1} \\
&\geqslant \sum_{j=1}^{N+1} |f(\tau_{4j+k})| \tau_{4j+k} \\
&\quad - \delta^{-1} \int_{0+}^{1} (|f(t)| + 1)\, dt
\end{aligned}$$

$\to \infty$ as $N \to \infty$, which contradicts (iii) above. This completes the proof of Theorem 5.

IV. Proof of Theorems 6 and 7

Proof of Theorem 6. (i) Suppose $S = S_g$ for some g in G. If $f \in S$, then $|f| \in S$. Let h denote the inverse function of g. Then the mapping $F \to F(h)\, h'$ maps the set of simply integrable functions onto S. If F, with domain $[0, \infty)$, is simply integrable, then $-|F(h)h'| \in S$; hence $-|F(h(g))| \cdot |h'(g)| \times (h'(g))^{-1} = |F|$ is simply integrable. Since the simple integral is not an absolute integral [1, p. 7], we have a contradiction.

(ii) Let F be improperly Riemann integrable on $[0, \infty)$ but not simply integrable (e.g., $F(x) = (\sin x)/x$, $F(0) = 1$; [1, p. 9]). Then $F(h)h'$ is improperly Riemann integrable on $(0, 1]$ but $F(h)h' \notin S_g$.

(iii) We shall prove this statement in several steps. $\int_0^\infty$ will denote improper Riemann integral on $[0, \infty)$.

(a) Given p, $0 < p < \infty$, there exists a function F_1, positive and continuous on $[0, \infty)$, and simply integrable, for which $\int_0^\infty F_1(x)\,dx = p$; and there exists a function F_2, positive and continuous on $[0, \infty)$, but not simply integrable, for which $\int_0^\infty F_2(x)\,dx = p$. Indeed, one can take, e.g., $F_1(x) = 2p/[\pi(x^2+1)]$ (cf. [2, p. 931]). As to F_2, by Theorem 5, it suffices to show that there exists a function f_p, positive, continuous, and improperly Riemann integrable on $(0, 1]$, with $\int_{0+}^1 f_p(x)\,dx = p$, which is not dominantly integrable. Obviously it is enough to show the existence of f_p for one (any) $p \in (0, \infty)$. Such a function is g, with domain $(0, 1]$, where $g(t) = 1$ off of the intervals $[2^{-n} - 4^{-n}, 2^{-n} + 4^{-n}]$, $n = 2, 3, \ldots$; on each interval $[2^{-n}, 2^{-n} + 4^{-n}]$, $n = 2, 3, \ldots$, $g(t) = 1 + 2^n - 8^n(t - 2^{-n})$; and on each interval $[2^{-n} - 4^{-n}, 2^{-n})$, $n = 2, 3, \ldots$, $g(t) = 1 + 8^n(t - 2^{-n} + 4^{-n})$. Clearly g is positive, continuous, and improperly Riemann integrable on $(0, 1]$. Since (again with $\hat{g}(t) = \sup_{t \leqslant x \leqslant 1} |g(x)|$, $0 < t \leqslant 1$), if $2^{-n-1} < t \leqslant 2^{-n}$ for some integer $n \geqslant 2$, then $\hat{g}(t) \geqslant g(2^{-n}) > 2^n > (2t)^{-1}$, we have $\hat{g}(t) > (2t)^{-1}$ for all t in $(0, 1/4]$, and hence (see Corollary 2 and its proof) g is not dominantly integrable.

(b) Given f_1, f_2, positive and continuous on $[0, \infty)$, with $\int_0^\infty f_1(x)\,dx = \int_0^\infty f_2(x)\,dx$, there exist g_1 and g_2 in G such that $f_1(h_1)h_1' = f_2(h_2)h_2'$, where h_k is the inverse function of g_k $(k = 1, 2)$. ($h_1'(1)$, $h_2'(1)$ are left-hand derivatives, $g_1'(0)$, $g_2'(0)$, $(d/dx)[g_1(x)]^{1/2}|_{x=0}$ and $(d/dx)[g_2(x)]^{1/2}|_{x=0}$ are right-hand.)

In fact, for $k = 1, 2$, set

$$g_k(x) = \left[1 - \frac{\int_0^x f_k(t)\,dt}{\int_0^\infty f_k(t)\,dt}\right]^2$$

so that, throughout $[0, \infty)$,

$$g_k'(x) = -2\left[1 - \frac{\int_0^x f_k(t)\,dt}{\int_0^\infty f_k(t)\,dt}\right]\frac{f_k(x)}{\int_0^\infty f_k(t)\,dt},$$

$$\frac{d}{dx}[g_k(x)]^{1/2} = \frac{g_k'(x)}{2[g_k(x)]^{1/2}} = -\frac{f_k(x)}{\int_0^\infty f_k(t)\,dt}.$$

Hence, throughout (0, 1],

$$f_1(h_1(x))\,h_1'(x) = \frac{f_1(h_1(x))}{g_1'(h_1(x))} = -\frac{\int_0^\infty f_1(t)\,dt}{2[g_1(h_1(x))]^{1/2}} = -\frac{\int_0^\infty f_1(t)\,dt}{2x^{1/2}}$$

$$= -\frac{\int_0^\infty f_2(t)\,dt}{2x^{1/2}} = f_2(h_2(x))\,h_2'(x).$$

(c) Given F, positive and continuous on $[0, \infty)$, and related to f of (iii) by

$$\int_0^\infty F(x)\,dx = \int_{0+}^1 f(x)\,dx,$$

there exists $g \in G$ such that $-f(g(x))\,g'(x) = F(x)$ throughout $[0, \infty)$. Indeed, let w be an arbitrary function in G (e.g., $(1+x)^{-1}$), and set

$$f_1 = -f(w)\,w', \qquad f_2 = F.$$

Then $\int_0^\infty f_1(x)\,dx = \int_0^\infty f_2(x)\,dx$. By (b), there exist g_1, g_2 in G such that $f_1(h_1)\,h_1' = f_2(h_2)\,h_2'$, where h_k is the inverse function of g_k $(k = 1, 2)$. Thus, $-f(w(h_1))\,w'(h_1)\,h_1' = F(h_2)\,h_2'$, and hence, throughout $[0, \infty)$,

$$-f(w(h_1(g_2(x))))\,w'(h_1(g_2(x)))\,h_1'(g_2(x))\,g_2'(x) = F(h_2(g_2(x))),$$

namely, $-f(g(x))\,g'(x) = F(x)$, where $g = w(h_1(g_2))) \in G$. We can now prove (iii). Set $p = \int_{0+}^1 f(x)\,dx$. Applying (c) to $F = F_1$ of (a), we obtain that there exists $g \in G$ for which $f \in S_g$. Applying (c) to $F = F_2$ of (a), we get that, for some other $g \in G$, f does not belong to S_g.

Proof of Theorem 7. For $n = 0, 1,\ldots,$ define $g_1(x)$ on $[7n, 7(n+1))$ as follows. On $[7n, 7n+1]$, $g_1(x) = e^{-x+6n}$. On $[7n+1, 7n+2]$, g_1 is quadratic; $g_1'(7n+1) = -e^{-n-1}$, and $\lim_{x\to 7n+2-} g_1'(x) = 0$. On $(7n+2, 7n+4]$, $g_1(x) = g_1(2(7n+2) - x)$. On $[7n+4, 7n+5]$, $g_1(x)$ is quadratic; $g_1'(7n+4) = e^{-n}$, and $\lim_{x\to 7n+5-} g_1'(x) = 0$. On $(7n+5, 7(n+1))$, $g_1(x) = g_1(2(7n+5) - x)$. This defines g_1 on $[0, \infty)$; also $g_1(6) = 1$, and g_1' is continuous in $(0, \infty)$. Set $g(x) \equiv g_1(x+6)$.

Assume that $f(g)\,g'$ is simply integrable. Then, [1], its ϵ-variation $V(\epsilon)$ is finite for each $\epsilon > 0$.

For $n = 1, 2,\ldots,$ let

$$S_n = \sup_{\substack{7n\leqslant x\leqslant 7n+1\\ 7n+3\leqslant y\leqslant 7n+4}} |f(g_1(x))\,g_1'(x) - f(g_1(y))\,g_1'(y)|\ (\leqslant V(2) < \infty),$$

and let x_n, y_n satisfy

$$7n \leqslant x_n \leqslant 7n + 1,$$
$$7n + 3 \leqslant y_n \leqslant 7n + 4,$$
$$|f(g_1(x_n))\, g_1'(x_n) - f(g_1(y_n))\, g_1'(y_n)| \geqslant (1/2)\, S_n$$

so that

$$\begin{aligned}
&|f(g_1(x_n))\, g_1'(x_n) - f(g_1(y_n))\, g_1'(y_n)| \\
&\quad \geqslant (1/2) \sup_{7n\leqslant x\leqslant 7n+1} |f(g_1(x))\, g_1'(x) \\
&\qquad - f(g_1(2(7n+2) - x))\, g_1'(2(7n+2) - x)| \\
&\quad = \sup_{7n\leqslant x\leqslant 7n+1} |f(g_1(x))\, g_1'(x)| \geqslant e^{-n-1} \sup\{|f(x)| : e^{-n-1} \leqslant x \leqslant e^{-n}\}.
\end{aligned}$$

Set $\xi_n = x_n - 6$, $\eta_n = y_n - 6$, $n = 1, 2, \ldots$. Then $1 \leqslant \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots$; $\eta_n - \xi_n \geqslant 2$, $\xi_{n+1} - \eta_n \geqslant 3$; $n = 1, 2, \ldots$. We have

$$\begin{aligned}
V(2) &\geqslant \sum_{n=1}^{\infty} |f(g(\eta_n))\, g'(\eta_n) - f(g(\xi_n))\, g'(\xi_n)| \\
&= \sum_{n=1}^{\infty} |f(g_1(y_n))\, g_1'(y_n) - f(g_1(x_n))\, g_1'(x_n)| \\
&\geqslant \sum_{n=1}^{\infty} e^{-n-1} \sup\{|f(x)| : e^{-n-1} \leqslant x \leqslant e^{-n}\}.
\end{aligned}$$

Set $s_k = \sup\{|f(x)|: e^{-k-1} \leqslant x \leqslant e^{-k}\}$, $k = 1, 2, \ldots$. Then for $N = 1, 2, \ldots$,

$$\sum_{n=1}^{N} \left(\sum_{k=1}^{n} s_k\right)(e^{-n} - e^{-n-1}) = \left(\sum_{n=1}^{N} s_n e^{-n}\right) - \left(\sum_{k=1}^{N} s_k\right) e^{-N-1}.$$

Because of the convergence of the last infinite series, $e^{-n-2} s_{n+1} \to 0$. For $\mu = 0, 1, 2, \ldots$, $0 \leqslant \nu \leqslant \mu$, let $a_{\mu,\nu} = e^{\nu-\mu}$, so that for every $p \geqslant 0$, $a_{n,p} = e^{p-n} \to 0$ as $n \to \infty$. Also, for every $n \geqslant 0$,

$$\sum_{k=0}^{n} |a_{n,k}| = e^{-n}(e^{n+1} - 1)/(e - 1) < e/(e - 1).$$

Hence [9, p. 72] $(\sum_{k=1}^{n+1} s_k)\, e^{-n-2} = \sum_{k=0}^{n} e^{k-n} e^{-k-2} s_{k+1} \to 0$, so that $(\sum_{k=1}^{N} s_k)\, e^{-N-1} \to 0$, and

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} s_k\right)(e^{-n} - e^{-n-1}) = \sum_{n=1}^{\infty} s_n e^{-n}.$$

Hence

$$\infty > V(2) \geqslant e^{-1} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} s_k \right) (e^{-n} - e^{-n-1})$$

$$\geqslant e^{-1} \sum_{n=1}^{\infty} [\sup\{|f(x)| : e^{-n-1} \leqslant x \leqslant e^{-1}\}](e^{-n} - e^{-n-1})$$

$$\geqslant e^{-1} \sum_{n=1}^{\infty} [\hat{f}(e^{-n-1}) - \hat{f}(e^{-1})](e^{-n} - e^{-n-1}).$$

The convergence of $\sum_{n=1}^{\infty} \hat{f}(e^{-n-1})(e^{-n} - e^{-n-1})$ implies that of $\int_{0+}^{1} \hat{f}(x)\, dx$. Thus, f is dominated in $(0, 1]$ by the monotone decreasing function $\hat{f}$, which is improperly Riemann integrable on $(0, 1]$. By Theorem 3, to show dominant integrability of f it suffices to prove that it is Riemann integrable on each $[e^{-n-1}, e^{-n}]$, $n = 0, 1,\ldots$. Choose such an n.

Since $f(g)\, g'$ is simply integrable, it is improperly Riemann integrable on $[0, \infty)$, and hence, Riemann integrable on $[7n, 7n + 1]$. Observe that $g' \neq 0$ throughout $[7n, 7n + 1]$. Let h be the inverse function of the restriction of g to $[7n, 7n + 1]$, so that $h(e^{-n-1}) = 7n + 1$, $h(e^{-n}) = 7n$. Then the substitution $x = h(t)$ in $\int_{7n}^{7n+1} f(g(x))\, g'(x)\, dx$ shows the Riemann integrability of f on $[e^{-n-1}, e^{-n}]$.

Conversely, suppose f is dominantly integrable. By Theorem 1, and the fact that g is strictly monotone on each $[k - 1, k]$, $k = 1, 2,\ldots$, we have, for every $T \geqslant 1$,

$$\left| \int_0^T f(g(x))\, g'(x)\, dx + \int_{0+}^{1} f(x)\, dx \right|$$

$$= \left| \left[\sum_{k=1}^{[T]} \int_{k-1}^{k} f(g(x))\, g'(x)\, dx \right] + \int_{[T]}^{T} f(g(x))\, g'(x)\, dx + \int_{0+}^{1} f(x)\, dx \right|$$

$$= \left| - \int_{g(T)}^{1} f(x)\, dx + \int_{0+}^{1} f(x)\, dx \right|$$

$$= \left| \int_{0+}^{g(T)} f(x)\, dx \right| \to 0 \qquad \text{as} \quad T \to \infty,$$

since $\lim_{T\to\infty} g(T) = 0$. ($[T]$ means the largest integer $\leqslant T$). By [1, Theorem 3], to prove simple integrability of $f(g)\, g'$ it suffices to show it is of bounded coarse variation.

Let $\epsilon > 0$. We shall show that if $0 \leqslant \xi_0 < \xi_1 < \cdots < \xi_N$, and $\min_{1\leqslant r\leqslant N} (\xi_r - \xi_{r-1}) \geqslant \epsilon$, then $\Delta = \sum_{r=1}^{N} |f(g(\xi_r))\, g'(\xi_r) - f(g(\xi_{r-1}))\, g'(\xi_{r-1})| \leqslant$

$14e^2(1+\epsilon^{-1})\int_{0+}^{1/e}\hat{f}(x)\,dx$ (the integral converging by Corollary 2 and its proof).

Now

$$\Delta \leqslant 2\sum_{r=0}^{N}|f(g(\xi_r))\,g'(\xi_r)| = 2\sum_{n=0}^{\infty}\sum_{\substack{0\leqslant r\leqslant N\\ e^{-n-1}<g(\xi_r)\leqslant e^{-n}}}|f(g(\xi_r))\,g'(\xi_r)|$$

(an "empty" sum means 0). For each n ($=0,1,2,\ldots$) there are seven intervals (or less) of the form $[k-1,k]$ ($k=1,2,\ldots$) whose union contains every $\xi\geqslant 0$ with $e^{-n-1}<g(\xi)\leqslant e^{-n}$; also, for every such ξ, $|g'(\xi)|\leqslant e^{-n}$. Hence

$$\Delta \leqslant 2\sum_{n=0}^{\infty}\hat{f}(e^{-n-1})\,e^{-n}\sum_{\substack{0\leqslant r\leqslant N\\ e^{-n-1}<g(\xi_r)\leqslant e^{-n}}}1.$$

For $k=1,2,\ldots$, there are at most $1+\epsilon^{-1}$ points ξ_r in $[k-1,k]$. Hence

$$\begin{aligned}\Delta &\leqslant 14(1+\epsilon^{-1})\sum_{n=0}^{\infty}\hat{f}(e^{-n-1})\,e^{-n}\\ &\leqslant 14e^2(1+\epsilon^{-1})\sum_{n=0}^{\infty}\hat{f}(e^{-n-1})(e^{-n-1}-e^{-n-2})\\ &\leqslant 14e^2(1+\epsilon^{-1})\int_{0+}^{1/e}\hat{f}(x)\,dx.\end{aligned}$$

References

1. S. Haber and O. Shisha, Improper integrals, simple integrals, and numerical quadrature, *J. Approximation Theory* **11** (1974), 1–15.
2. S. Haber and O. Shisha, An integral related to numerical integration, *Bull. Amer. Math. Soc.* **79** (1973), 930–932.
3. P. J. Davis and P. Rabinowitz, Ignoring the singularity in approximate integration, *J. SIAM Numer. Anal. Ser. B.* **2** (1965), 367–382.
4. P. Rabinowitz, Gaussian integration in the presence of a singularity, *SIAM J. Numer. Anal.* **4** (1967), 191–201.
5. W. Gautschi, Numerical quadrature in the presence of a singularity, *SIAM J. Numer. Anal.* **4** (1967), 357–362.
6. R. K. Miller, On ignoring the singularity in numerical quadrature, *Math. Comput.* **25** (1971), 521–532.
7. A. Feldstein and R. K. Miller, Error bounds for compound quadrature of weakly singular integrals, *Math. Comput.* **25** (1971), 505–520.
8. C. F. Osgood and O. Shisha, Numerical quadrature of improper integrals and the dominated integral, *J. Approximation Theory*, to appear.
9. K. Knopp, "Theory and Application of Infinite Series," Blackie, London, 1949.